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NOTE

On the Number of Permutations Avoiding a Given Pattern

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Let $\sigma \in S_k$ and $\tau \in S_n$ be permutations. We say τ contains σ if there exist $1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant n$ such that $\tau(x_i) < \tau(x_j)$ if and only if $\sigma(i) < \sigma(j)$. If τ does not contain σ we say τ avoids σ. Let $F(n, \sigma) = |\{\tau \in S_n \mid \tau \text{ avoids } \sigma\}|$. Stanley and Wilf conjectured that for any $\sigma \in S_k$ there exists a constant $c = c(\sigma)$ such that $F(n, \sigma) \leqslant c^n$ for all n. Here we prove the following weaker statement: For every fixed $\sigma \in S_k$, $F(n, \sigma) \leqslant c^{n\gamma^*(n)}$, where $c = c(\sigma)$ and $\gamma^*(n)$ is an extremely slow growing function, related to the Ackermann hierarchy. © 2000 Academic Press

1. INTRODUCTION

Let $\sigma \in S_k$ and $\tau \in S_n$ be permutations. We say τ contains σ, and denote this by $\sigma < \tau$, if there exist $1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant n$ such that $\tau(x_i) < \tau(x_j)$ if and only if $\sigma(i) < \sigma(j)$. If τ does not contain σ we say τ avoids σ. Thus,

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(representing σ by $\sigma(1), \sigma(2), ..., \sigma(k)$) 1523647 contains 132 but avoids 321. Let

$$F(n, \sigma) = |\{\tau \in S_n \mid \tau \text{ avoids } \sigma\}|.$$

For any $\sigma \in S_3$ it is known (see, e.g., [9]) that $F(n, \sigma) = \binom{2n}{n}/(n+1)$. Bóna [2] calculated the precise value of $F(n, \sigma)$ for $\sigma = 1342$, and obtained exponential upper bounds for $F(n, \sigma)$ for all $\sigma \in S_4$ [1]. When σ is the identity of S_k, $F(n, \sigma)$ is the number of n-permutations with no increasing subsequence of length k. Such permutations can be partitioned into $k-1$ monotone subsequences, and hence one can show that the number of them is less than $(k-1)^{2n}$. The exact asymptotics for this case is also known [7]. The following conjecture of Stanley and Wilf is open (cf. [1, 3]):

Conjecture 1.1. For every σ there exists a constant $c = c(\sigma)$ such that $F(n, \sigma) \leqslant c^n$ for every n. They also suggested a stronger conjecture, namely, that for every fixed σ the limit, as n tends to infinity, of $(F(n, \sigma))^{1/n}$ exists and is finite and positive.

Conjecture 1.1 is known to be true in many special cases, see [3] and its references. In this note we prove a slightly weaker result, as follows, and prove the conjecture for a certain class of permutations.

First let us define some slowly growing functions. Let $\alpha(n)$ be the inverse of the Ackermann function, defined as follows.

For any function f, put $f_1(n) = f(n)$, $f_i(n) = f(f_{i-1}(n))$. The family of functions $A^{(k)}(n)$ is defined by induction as follows. $A^{(k)}(1) = 2$, $A^{(1)}(n) = 2n$ and $A^{(k)}(n) = (A^{(k-1)})_n\,(1)$. Then

$$\alpha(n) = \min\{s \geqslant 1 \mid A^{(s)}(s) \geqslant n\}.$$

As k is fixed throughout this paper define β

$$\beta(m) = 2k2^{k^2-4}(10k)^{2(\alpha(m))^{k^2-4} + 8(\alpha(m))^{k^2-5}}.$$

For an integer $n > \beta(1)$ let $m = m(n)$ be defined as the largest integer such that $m\beta(m) \leqslant n$, (for $n \leqslant \beta(1)$ put $m(n) = 1$.) Define $\gamma(n) = \lceil n/m \rceil$. Finally, define $\gamma^*(n)$ to be the smallest integer t such that $\gamma_t(n) \leqslant 2\beta(2)$. Note that $\gamma^*(n)$ is an extremely slow growing function, and (as k is fixed) it is much smaller than $\alpha(n)$ for all sufficiently large n.

Our main result is the following.

THEOREM 1.2. *There exists a constant $c = c(k)$ such that for every $\sigma \in S_k$ $F(n, \sigma) \leqslant c^{n\gamma^*(n)}$ for every n.*

The proof of this theorem appears in the next section. In Section 3 we prove that Conjecture 1.1 holds for every permutation which consists of an increasing subsequence followed by a decreasing one, or vice versa.

2. THE PROOF

Before presenting the proof here are some definitions we need. To avoid excessive notation, let $\sigma \in S_k$ be a fixed permutation throughout the rest of this note. For a vector $t \in \{1, ..., m\}^n$ we wish to distinguish between containing a given permutation (or pattern) and containing a given subword. We say that t contains the pattern σ and denote this by $\sigma < t$ exactly as we did for a permutation in S_n: $\sigma < t$ if there exist indices $1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant n$ such that $t_{x_i} < t_{x_j}$ if and only if $\sigma(i) < \sigma(j)$. Note here that all inequalities are strict. For $y \in \{1, ..., k\}^r$ with $r \leqslant n$ we say t contains the subword y if there exist indices $1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant n$ such that $t_{x_i} = t_{x_j}$ if and only if $y_i = y_j$. Thus, for example, 143643 does not contain the pattern 1234 but does contain the subword 1234 and also the subword 1212.

Recalling that σ is fixed we let $F(n) = F(n, \sigma)$.

Let $A(n, m) = |\{t \in \{1, ..., m\}^n \mid t \text{ avoids } \sigma\}|$. We say a word $t \in \{1, ..., m\}^n$ is k-regular if $t_i = t_j$, $i \neq j$ implies $|i - j| \geqslant k$.

For a given word $y \in \{1, ..., k\}^r$ let

$$\ell(y, m) = \max\{n \mid \exists t \in \{1, ..., m\}^n, t \textit{ is } k\text{-regular and avoids } y\}.$$

The question of determining $\ell(y, m)$ when y is of the form *ababa* and some of its variations is that of finding the maximum possible length of Davenport–Schinzel sequences, and has received a lot of attention (see [8] and its many references). Here we use the following theorem about generalized Davenport–Schinzel sequences due to Klazar:

THEOREM 2.1 [5]. *For every k and r and every word $y \in \{1, ..., k\}^r$*

$$\ell(y, m) \leqslant m 2k 2^{r-4} (10k)^{2(\alpha(m))^{r-4} + 8(\alpha(m))^{r-5}},$$

where α is the inverse of the Ackermann function.

We use $\beta(m)$ to denote the function multiplying m in this bound for $r = k^2$.

Our two main lemmas are the following.

LEMMA 2.2. *For any $0 < m < n$*

$$F(n) \leqslant F(\lceil n/m \rceil)^m \, A(n, m).$$

LEMMA 2.3. *If $m\beta(m) \leqslant n$ then*

$$A(n, m) \leqslant (8k^4)^n.$$

Before proving these lemmas let us see how the proof of Theorem 1.2 follows:

Proof of Theorem 1.2. Recall that for an integer $n > \beta(1)$ we define $m(n)$ as the maximal integer such that $m\beta(m) \leqslant n$ and $\gamma(n) = \lceil n/m \rceil$. Let $n_0 = n$, $n_i = \gamma(n_{i-1})$ for $i > 0$, and $m_i = m(n_i)$. Combining the two lemmas we get

$$F(n_i) \leqslant F(n_{i+1})^{m_i}\, (8k^4)^{n_i}.$$

It is more convenient to look at the function $G(n) = F(n)^{1/n}$. For this function we get the recurrence

$$G(n_i) \leqslant G(n_{i+1})^{m_i/n_i \lceil n_i/m_i \rceil}\, 8k^4.$$

Note that $(m_i/n_i) \lceil (n_i/m_i) \rceil \leqslant 1 + 1/\beta(m_i)$. Therefore using the above estimate for n_0 and iterating we have

$$\begin{aligned} G(n_0) &\leqslant G(n_1)^{1 + 1/\beta(m_o)} 8k^4 \\ &\leqslant G(n_2)^{(1 + 1/\beta(m_0)) \cdot (1 + 1/\beta(m_1))} (8k^4)^{1 + (1 + 1/\beta(m_o))} \\ &\leqslant \cdots \leqslant c'(k) (8k^4)^{1 + (1 \cdot (1 + 1/\beta(m_0))) + (1 \cdot (1 + 1/\beta(m_0)) \cdot (1 + 1/\beta(m_1)))\ldots} \\ &\leqslant c(k)^{\gamma^*(n_0)}. \end{aligned}$$

We have used here the fact that the product $1 \cdot (1 + 1/\beta(m_0) \cdot (1 + 1/\beta(m_1))\ldots$ is bounded for every integer m_0, and the fact that $G(l) \leqslant c'(k)$ for all $l \leqslant 2\beta(2)$.

It remains to prove the two lemmas.

Proof of Lemma 2.2. Any permutation in S_n that avoids σ can be achieved uniquely in the following way: take a word $t \in \{1, \ldots, m\}^n$ that (disregarding questions of divisibility) has exactly n/m copies of each letter and avoids σ. There are at most $A(n, m)$ of these. Now substitute a permutation of the numbers $1, \ldots, n/m$ which avoids σ for all the 1's, a permutation of $n/m + 1, \ldots, 2n/m$ which avoids σ for the 2's etc. There are at most $F(n/m)^m$ ways to choose these permutations. This, and the simple fact that $F(n)$ is monotone in n, yields the desired estimate in the case where m does not necessarily divide n. ∎

Proof of Lemma 2.3. The lemma follows readily by induction and by combining the two estimates

$$A(n, m) \leqslant k^n A(m\beta(m), m) \tag{1}$$

and

$$A(m\beta(m), m) \leqslant A\left(m\beta(m), \left\lceil\frac{m}{2}\right\rceil\right) 2^{m\beta(m)}. \tag{2}$$

Indeed, by repeatedly applying (2) and (1) we conclude that

$$\begin{aligned} A(m\beta(m), m) &\leqslant (2k)^{m\beta(m)} A\left(\left\lceil\frac{m}{2}\right\rceil \beta\left(\left\lceil\frac{m}{2}\right\rceil\right), \left\lceil\frac{m}{2}\right\rceil\right) \\ &\leqslant (2k)^{m\beta(m)+\lceil m/2\rceil \beta(\lceil m/2\rceil)+\cdots}. \end{aligned}$$

Since $m\beta(m) \leqslant n$, another application of (1) supplies the desired result (with room to spare).

Let us first prove (1): obviously, any sequence $t = t_1, ..., t_n$ that avoids a permutation in S_k must also avoid the subword

$$\overbrace{a_1, a_2, ..., a_k, a_1, a_2, ..., a_k, ..., a_1, ..., a_k}^{k \text{ times}}.$$

Let $t \in \{1, ..., m\}^n$ be such a word. By Theorem 2.1 any k-regular subsequence of t is of length at most $m\beta(m)$. The following procedure gives a label from $\{0..., k-1\}$ to each letter, and partitions t into two subsequences t_1 and t_2. The first one, t_1 which we call the "regular" subsequence, will be k-regular. The procedure is as follows: we start with two empty sequences t_1 and t_2 and refer to the letters in t_1 as the regular letters. Then we scan the letters of t sequentially, and whenever we encounter a letter different from the last $k-1$ regular letters (or from all elements of t_1, at the stage when there are less than $k-1$ of them), we declare it to be regular, append it to the end of t_1 and give it the label 0. If it is equal to one of the $k-1$ previous regular letters we give it a label between 1 and $k-1$ to designate which it was equal to and append it to t_2. Since the length of the regular subsequence t_1 is at most $m\beta(m)$ there are at most $A(m\beta(m), m)$ possibilities for the actual sequence t_1. The number of choices for the ordered set of labels is k^n. Moreover, the sequence t_1, and the ordered sequence of labels, determine t uniquely. This proves the inequality (1).

The proof of (2) is similar to the proof of Lemma 2.2. Taking a pattern-avoiding word of length $m\beta(m)$ using the letters $\{1 \cdots m\}$ we identify the letters in pairs: 1 with 2, 3 with 4, etc. The resulting word is composed of $\lceil m/2\rceil$ letters. This contributes the $A(m\beta(m), \lceil m/2\rceil)$ factor. The $2^{m\beta(m)}$ factor comes from the possibilities of decoding such a word back to the original one. This completes the proof of Lemma 2.3 and with it the proof of the theorem. ∎

3. CASES IN WHICH THERE IS AN EXPONENTIAL UPPER BOUND

As we mentioned in the Introduction it was known that Conjecture 1.1 holds for permutations that are either an increasing sequence (the identity) or a decreasing sequence. Bóna also proved the conjecture in the case of "layered" permutations, where the permutation is a series of monotone increasing (decreasing) subsequences, and the members of each subsequence are smaller (larger, respectively) than those of the previous subsequence. Using the same technique as in the previous section and another work of Klazar and Valtr from the theory of Davenport Schinzel sequences we can prove the conjecture for another class of permutations. Let

$$A_{uu}(k) = \{\sigma \in S_k \mid \sigma \text{ is the concatenation of two increasing subsequences}\}$$

$$A_{ud}(k) = \{\sigma \in S_k \mid \sigma \text{ consists of an increasing subsequence followed by a decreasing one}\}$$

and define A_{du}, A_{dd}, A_{dud} and A_{udu} similarly. For a pair of permutations σ_1, σ_2 let

$$F(n, \sigma_1, \sigma_2) = |\{\tau \in S_n \mid \tau \textit{ avoids both } \sigma_1 \text{ and } \sigma_2\}|.$$

THEOREM 3.1. *There exists a constant $c = c(k)$ such that for every n and every permutation $\sigma \in (A_{ud}(k) \cup A_{du}(k))$, $F(n, \sigma) \leqslant c^n$.*

Furthermore, for every pair of permutations $\sigma_1 \in A_{udu}(k)$ and $\sigma_2 \in A_{dud}(k)$ $F(n, \sigma_1, \sigma_2) \leqslant c^n$.

The key to the proof is the following observation: For a permutation $\sigma \in S_k$ and an integer r define $\ell_r(\sigma, m)$ in a way similar to the definition for the case of a forbidden pattern,

$$\ell_r(\sigma, m) = \max\{n \mid \exists t \in \{1, ..., m\}^n, t \textit{ is } r\text{-regular and avoids } \sigma\}.$$

Where we used the function $m\beta(m)$ in Lemma 2.3 in the proof of Theorem 1.2 what we actually needed was $\ell_r(\sigma, m)$. If for a certain permutation σ and for some r bounded by a function of k one can show that $\ell_r(\sigma, m)$ is actually linear in m, the same proof gives us that $F(n) = F(\sigma, n) \leqslant c(k)^n$. Thus Theorem 3.1 follows from the following lemma:

LEMMA 3.2. *There exists a function $c(k)$ such that for any $(k-1)^2+1$-regular word $t \in \{1, ..., n\}^{c(k)\,n}$ the following three conditions hold:*

- *t contains every permutation in $A_{ud}(k)$.*
- *t contains every permutation in $A_{du}(k)$.*

- *t either contains every permutation in $A_{udu}(k)$ or every permutation in $A_{dud}(k)$.*

It is worth noting that the assumption that t is $(k-1)^2+1$-regular can be replaced by the weaker one that it is k-regular, but since for our purpose here the above version suffices we omit the (simple) argument showing that the two versions are equivalent.

The last lemma follows from the following two results. The first is due to Klazar and Valtr:

THEOREM 3.3 [6]. *Let $a_1, ..., a_r$ be symbols. Consider the word*

$$y = a_1 a_2 \cdots a_{r-1} a_r a_{r-1} a_{r-2} \cdots a_2 a_1 a_2 a_3 \cdots a_r.$$

Then $\ell(y, m) = O(m)$

Also, we need the well known

LEMMA 3.4 (Erdős and Szekeres [4]). *Any sequence of numbers of length $(k-1)^2+1$ contains a monotone subsequence of length k.*

Deducing Lemma 3.2 from the above is not difficult. By taking $r = (k-1)^2+1$ in Theorem 3.3 we conclude that there is a $c = c(k)$ such that any $(k-1)^2+1$-regular word of length cn over $\{1, 2, ..., n\}$ contains the word y. The result now follows since by Lemma 3.4 the sequence $a_1 a_2 \cdots a_r$ contains either an increasing or a decreasing subsequence of length k.

It follows from the above discussion that conjecture 1.1 would follow if one could prove a linear bound for $\ell_k(\sigma, n)$ for any $\sigma \in S_k$ (although the opposite implication is not clear). This seems like an interesting question in its own right:

QUESTION 3.5. *Is it true that for every permutation $\sigma \in S_k$ there exists $c(\sigma)$ such that $\ell_k(\sigma, n) \leqslant cn$ for all n?*

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